ON SCATTERED POSETS WITH FINITE DIMENSION

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To the memory of Eric C.Milner (1928-1997)

Abstract. We discuss a possible characterization, by means of forbidden configurations, of posets which are embeddable in a product of finitely many scattered chains.

Introduction and presentation of the results

A fundamental result, due to Szpilrajn [26], states that every order on a set is the intersection of a family of linear orders on this set. The *dimension* of the order, also called the dimension of the ordered set, is then defined as the minimum cardinality of such a family (Dushnik, Miller [11]). Specialization of Szpilrajn's result to several types of orders have been studied [3]. An ordered set (in short poset), or its order, is *scattered* if it does not contain a subset which is ordered as the chain η of rational numbers. Bonnet and Pouzet [2] proved that *a poset is scattered if and only if the order is the intersection of scattered linear orders.* It turns out that there are scattered posets whose order is the intersection of finitely many linear orders but which cannot be the intersection of finitely many scattered linear orders. We give nine examples in Theorem 1. This naturally leads to the following question:

Question 1. *If an order is the intersection of finitely many linear scattered orders, does this order the intersection of n many scattered linear orders, where n is the dimension of this order?*

We do not have the answer even for dimension two orders. We cannot even answer this:

Question 2. *If an order of dimension two is the intersection of three scattered linear orders, does this order the intersection of two scattered linear orders?*

Question 1 is a special instance of the following general qestion:

Given a positive integer n, which orders are intersection of at most n scattered linear orders?

We propose an approach based on the notion of obstruction.

Let n be a non negative integer; let $\mathcal{L}(n)$, resp. $\mathcal{L}_{\mathcal{S}}(n)$ be the class of posets P whose order is the the intersection of at most n linear orders, resp. at most n scattered linear orders. Set $\mathcal{L}(<\omega) := \bigcup_{n<\omega} \mathcal{L}(n)$ and $\mathcal{L}_{\mathcal{S}}(<\omega) := \bigcup_{n<\omega} \mathcal{L}_{\mathcal{S}}(n)$.

Date: December 9, 2008.

2000 *Mathematics Subject Classification.* 06 A06, 06 A15, 54G12.

Key words and phrases. Ordered sets, Scattered posets, Scattered topological spaces. Dushnik-Miller dimension.

Research done under the auspices of the CMCU Franco-Tunisien "Outils mathématiques pour l'informatique". This research was completed while the authors visited each others. Support provided by the university of Ottawa is gratefully acknowledged.

These four classes are *closed under embeddability*, that is if $\mathcal{C}$ is one of these classes, then for every poset $P \in \mathcal{C}$, a poset Q belongs to $\mathcal{C}$ whenever it is embeddable in P (that is Q is isomorphic to an induced subposet of P). Say that an *obstruction* to a class $\mathcal{C}$ as above is any poset not belonging to $\mathcal{C}$. Then such a class $\mathcal{C}$ can be characterized by obstructions, eg as the class of posets in which no obstruction to $\mathcal{C}$ is embeddable. But, it can be also characterized by means of smaller collections of obstructions. If $\mathcal{B}$ is a class of poset, denote by $Forb(\mathcal{B})$ the class of posets in which no member of $\mathcal{B}$ is embeddable.

With this terminology, we may ask:

Find $\mathcal{B}$ as simple as possible such that $\mathcal{L}_{\mathcal{S}}(n) = Forb(\mathcal{B})$.

The following question emerges immediately:

Question 3. *Is there a cardinal λ such that every obstruction to $\mathcal{L}_{\mathcal{S}}(n)$ contains an obstruction of size at most λ?*

As it can be easily seen, the existence of such a cardinal for an arbitrary class closed under embeddability follows readily from the *Vopěnka principle*, a strong set theoretical principle which could be inconsistent with usual set theoretical axioms. It implies the existence of large cardinal numbers (eg supercompact cardinals) and its consistency is implied by the existence of huge cardinals (see [15] pp. 413–415).

In the case of $\mathcal{L}_{\mathcal{S}}(n)$ we do not know if λ exists. In fact, *we conjecture that it exists and is countable.*

The same general question for $\mathcal{L}(n)$ has a simpler answer: each obstruction contains a finite one. Indeed, as it is well known, *a poset P belongs to $\mathcal{L}(n)$ whenever for every* finite *subset A of P the poset induced by P on A is also in $\mathcal{L}(n)$* (this striking fact is a consequence of the compactness theorem of first order logic - for a proof, see the survey [16]). Furthermore, if $Crit(\mathcal{L}(n))$ denotes the collection of minimal obstructions (that is the collection of finite posets Q whose dimension is larger than n, whereas every proper subposet has dimension at most n), then $\mathcal{L}(n) = Forb(Crit(\mathcal{L}(n)))$. Members of $Crit(\mathcal{L}(n))$ have dimension $n+1$; these posets are the so-called $n+1$*-irreducible posets* [28]. For $n = 1$, there is just one: the two element antichain. For $n = 2$, a complete description has been given by D.Kelly in 1972 (see [16]). For $n > 2$ a description seems to be hopeless; in fact, the problem to decide whether or not a finite poset belongs to $\mathcal{L}(n)$ is NP-complete. If $\mathcal{C} = \mathcal{L}(< \omega)$, every obstruction contains a countable one (this easily follows from the finitary result mentionned above), hence $\mathcal{L}(< \omega) = Forb(\mathcal{B})$ where $\mathcal{B}$ is a set of countable posets, each with a countable dimension. In terms of obstructions, Question 1 amounts to:

Question 4. *Is $Crit(\mathcal{L}(n))$ determines $\mathcal{L}_{\mathcal{S}}(n)$ within $\mathcal{L}_{\mathcal{S}}(< \omega)$?*

We rather consider the following:

Question 5. *Is $\mathcal{L}_{\mathcal{S}}(< \omega)$ can be determined within $\mathcal{L}(< \omega)$ by a finite set $\mathcal{B}_{\mathcal{S}}$ of obstructions?*

We provide ten examples of obstructions. All are countable and have dimension at most 3. In order to present these examples, we denote by P^* the dual of a poset P, we denote by $\check{P}$ the set P equipped with the strict order $<$. We denote by $B(\check{P})$ the poset defined as follows: the underlying set is $P \times \{0, 1\}$, the ordering defined by $(x, i) < (y, j)$ if $i < j$ and $x < y$. This poset is the *open split* of P. It

is clearly bipartite, moreover $B(\check{P}^*)$ is order-isomorphic to $B(\check{P})^*$. Let T_2 be the infinite binary tree and let $\Omega(\eta)$ be the infinite binary tree in which each level is totally ordered by an increasing way from the left to the right (see Figure 1 for an equivalent representation).

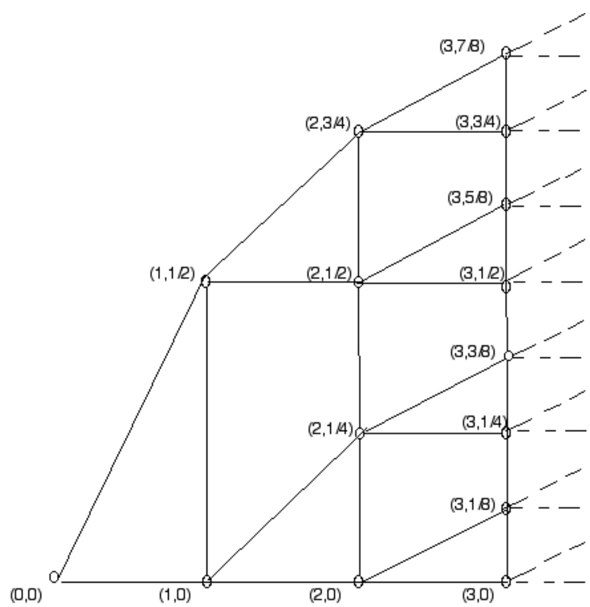

FIGURE 1. $\Omega(\eta)$

We prove:

Theorem 1. *A poset whose the order is the intersection of finitely many scattered linear orders contains no isomorphic copy of* η, T_2, $\Omega(\eta)$, $B(\check{\eta})$, $B(\check{T}_2)$, $B(\check{\Omega}(\eta))$ *and their dual.*

Since η and $B(\check{\eta})$ are self dual, this list contains only ten members. In fact, these members do not embed in each other (Lemma 19).

Problem 1. *Is this list determines the class* $\mathcal{L}_{\mathcal{S}}(<\omega)$ *of orders which are intersection of finitely many scattered linear orders within the class* $\mathcal{L}(<\omega)$ *of orders which are intersection of finitely many linear orders?*

The reader will notice that each of our obstructions distinct from η contains an infinite antichain. This is general. Indeed, if a poset P is scattered with no infinite antichain, each linear extension of the order on P is scattered ([2], see also [3]), hence if $dim(P) = n$, the order is the intersection of n scattered linear orders.

The occurence of open splits in Theorem 1 asks for an explanation. We present one, despite the fact that it is not fully satisfactory. It is based on the notion of split rather than open split. If P is a poset, the *split* of P is the poset $B(P)$ whose underlying set is $P \times \{0,1\}$ ordered by:

$$(x,i) < (y,j) \text{ if } \ x \leq y \ \text{ and } i < j.$$

We prove:

Theorem 2. *Let P be a poset. Then $P \in \mathcal{L}_{\mathcal{S}}(<\omega)$ if and only if $B(P) \in \mathcal{L}_{\mathcal{S}}(<\omega)$.*

The analogous equivalence with $B(\check{P})$ instead of $B(P)$ is in general false. But, if P is η, T_2, $\Omega(\eta)$ or their dual, $B(P)$ and $B(\check{P})$ can be embedded in each other (Lemma 17). Hence, in order to prove Theorem 1 it suffices to prove that η, T_2, $\Omega(\eta)$ and their dual are obstructions to $\mathcal{L}_S(<\omega)$ and to apply Theorem 2. In order to do that, we introduce a peculiar object: the topological closure $\overline{N(P)}$ in the powerset $\mathfrak{P}(P)$ of the MacNeille completion $N(P)$ of a poset P. As a poset, $\overline{N(P)}$ is an algebraic lattice.

We prove:

Theorem 3. *Let P be a poset and n be a positive integer. Then the following properties are equivalent:*

(i) The order on P is the intersection of n scattered linear orders;

(ii) $\overline{N(P)}$ is embeddable into a product of n scattered linear orders.

Moreover, if one of these conditions hold, $\overline{N(P)}$ is topologically scattered.

With this result at hand, in order to show that if P is η, T_2 or $\Omega(\eta)$, P is an obstruction, it suffices to observe that $\overline{N(P)}$ is not topologically scattered. We give the proof of this fact in Section 3.

Note that while $N(P)$ and $N(P^*)$ are dually isomorphic, $\overline{N(P)}$ and $\overline{N(P^*)}$ are not. Hence, one can be topologically scattered, whereas the other is not. For an example, $\overline{N(T_2^*)}$ is topologically scattered and $\overline{N(T_2)}$ is not.

Question 6. *If $dim(P) \leq n$ and both $\overline{N(P)}$ and $\overline{N(P^*)}$ are topologically scattered does the order on P is the intersection of n scattered linear orders?*

After such unsuccessful attempt of a description of $\mathcal{L}_{\mathcal{S}}(<\omega)$ by means of obstructions, we looked at subclasses $\mathcal{C}$ of $\mathcal{L}_{\mathcal{S}}(<\omega)$ such that every member of $\mathcal{L}_{\mathcal{S}}(<\omega)$ can be embedded in a member of $\mathcal{C}$. It turns out that the class of scattered distributive lattices of finite dimension has this property. In fact:

Theorem 4. *Let T be a distributive lattice. The following properties are equivalent:*

(i) The order on T is the intersection of n scattered linear orders.

(ii) T is isomorphic to a sublattice of a product of n scattered chains.

(iii) $dim(T) \leq n$ and T is order-scattered.

We also consider extensions of our initial question.

Instead of linear orders, we consider interval orders and instead of scattered linear orders, interval orders which can be represented as intervals of a scattered chain. Instead of ordered sets we consider incidence structures, we replace linear orders by Ferrers relations, we replace MacNeille completion by Galois lattices and scattered linear orders by Ferrers relations whose Galois lattice is scattered. We obtain an extension of Theorem 3 (see Theorem 8). From our study, it follows that a positive answer to our initial question implies a positive answer to the extensions we consider. The basic objets of our study are incidence structures and Galois lattices. One of our key result is a property of the topological closure of Galois lattices (Theorem 7) which refines Bouchet's Coding theorem ([4], see also [5], see Theorem 6).

To conclude, we mention a specialization of our question which comes from the following observation. All finite ordered sets of dimension 2 are obtained as follows. Let $\underline{n} := \{0, 2, ..., n-1\}, n \geq 2$ and C be the linear ordering $0 < 1 < \cdots < n-1$ on $\underline{n}$. Let σ be a permutation on $\underline{n}$, distinct of the identity map. Define the order

$\leq_\sigma$ on $\underline{n}$ by $x \leq_\sigma y$ if and only if $x \leq_C y$ and $\sigma(x) \leq_C \sigma(y)$. Let $P_\sigma := (\underline{n}, \leq_\sigma)$, and $C_\sigma := \{(x,y) : \sigma(x) \leq_C \sigma(y)\}$, then $\leq_\sigma$ is the intersection of C and C_σ. Thus P_σ has dimension 2. For infinite posets, even countable, the situation is quite different. Order which are intersection of two orders of type ω are close to finite orders. A characterization in terms of obstructions is included in the following:

Theorem 5. *An order on an infinite set is the intersection of n linear orders of type ω if and only if:*

(i) The order has dimension at most n.
(ii) The poset does not contain an infinite antichain, an infinite decreasing chain, the chain $\omega + 1$ and the direct sum $\omega \oplus 1$ of the chain ω with the one-element chain.

See Proposition 4.1 and Corollary 4.2 of [20].

A more general question is the following:

Question 7. *Given a positive integer n and an order type α. Which orders are the intersection of n linear orders of the same type α?*

More specifically

Question 8. *Characterize by means of obstructions the posets which are embeddable into posets whose order is the intersection of n linear orders of the same type α?*

This paper is composed as follows. Section 1 contains the definitions of the main notions with a development on incidence structures and Galois lattices and coding. It includes our refinement of Bouchet's Coding theorem, and also some basic facts on Ferrers relations, interval orders and dimension. Section 2 contains a discussion on the notions of scattered dimension, including Theorem 8. Sections 3 and 4 contains the proofs of the results presented above. Section 5 contains a characterization of orders which are intersection of two scattered linear orders (Theorem 13).

1. Ingredients

Our terminology follows [7] and [14]. Among set theoretical notations, we point out that if f is a map from a set E to a set F, and A is a subset of E, the set $\{f(x) : x \in A\}$, the *image* of A by f, is denoted by $f[A]$ rather than $f(A)$.

1.1. **Order, lattices and topology.** As usual, a *poset* is the pair P formed of a set E and an order ε on E. If the order is *linear*(or total), the poset is a *chain*. The *dual* of $P := (E, \varepsilon)$ is $P^* := (E, \varepsilon^{-1})$. If this causes no confusion, we will denote an order on E by the symbol $\leq$ and its complement by $\not\leq$; we will denote the equality relation by $=$ (and, when needed, by $\Delta_E := \{(x,x) : x \in E\}$), we identify P with E, writing $x \in P$ instead of $x \in E$. We will denote by $x \parallel_P y$ the fact that two elements x and y of P are incomparable. Given a poset $P := (E, \leq)$, a subset I of E is an *initial segment* (or is *closed downward*) if $x \leq y$ and $y \in I$ imply $x \in I$. Let X be a subset of E, we set:

$$\downarrow X := \{y \in E : y \leq x \text{ for some } x \in X\}. \tag{1}$$

This set is an initial segment, in fact the least initial segment containing X. We say that $\downarrow X$ is generated by X. If X contains only one element x, we write $\downarrow x$ instead of $\downarrow \{x\}$. An initial segment of this form is *principal*. We set $down(P) := \{\downarrow x : x \in P\}$.

We denote by $\mathbf{I}(P)$ the set of initial segments of P ordered by inclusion. For example, $\mathbf{I}((E, \Delta_E)) = \mathfrak{P}(E)$ the power set of E ordered by inclusion, whereas $\mathbf{I}((\mathbb{Q}, \leq))$ is the *Cantor chain*. We also denote by $\mathbf{I}_{<\omega}(P)$ the set of finitely generated initial segments of P ordered by inclusion. An *ideal* of P is a non empty initial segment I which is up-directed, that is every pair $x, y \in I$ has an upper bound $z \in I$. We denote by $\mathcal{J}(P)$ the set of ideals of P and by $\mathcal{J}^{\neg\downarrow}(P)$ the subset of non-principal ideals of P. Let $N(P)$ be the set made of intersections of principal initial segments of P. Ordered by inclusion, $N(P)$ is a complete lattice, called the *MacNeille completion of* P.

A *join-semilattice* is a poset P such that every two elements x, y have a least upper-bound, or join, denoted by $x \vee y$. If P has a least element, that we denote 0, this amounts to say that every finite subset of P has a join. An element a in a lattice L is *compact* if for every $A \subset L$, $a \leq \bigvee A$ implies $a \leq \bigvee A'$ for some finite subset A' of A. The lattice L is *compactly generated* if every element is a supremum of compact elements. A lattice is *algebraic* if it is complete and compactly generated. Algebraic lattices and join-semilattices with a least element are sides of the same coin. Indeed, the set $K(L)$ of compact elements of an algebraic lattice L is a join-semilattice with a least element and L is isomorphic to the set $\mathcal{J}(K(L))$ of ideals of $K(L)$, ordered by inclusion. Conversely, the set $\mathcal{J}(P)$ of ideals of a join-semilattice P having a least element, once ordered by inclusion, is an algebraic lattice, and the subset $K(\mathcal{J}(P))$ of its compact elements is isomorphic to P. We note that if P is an arbitrary poset, $\mathbf{I}(P)$ is an algebraic lattice and $K(\mathbf{I}(P)) = \mathbf{I}_{<\omega}(P)$. Hence, $\mathcal{J}(\mathbf{I}_{<\omega}(P))$ is order isomorphic to $\mathbf{I}(P)$. We also note that $\mathcal{J}(P)$ is the set of join-irreducible elements of $I(P)$; moreover, $\mathbf{I}_{<\omega}(\mathcal{J}(P))$ is order-isomorphic to $\mathbf{I}(P)$ whenever P has no infinite antichain.

Identifying the power set $\mathfrak{P}(E)$ of a set E with 2^E, we may view it as a topological space. A basis of open sets consists of subsets of the form $O(F, G) := \{X \in \mathfrak{P}(E) : F \subseteq X \text{ and } G \cap X = \emptyset\}$, where F, G are finite subsets of E. As it is customary, we denote by $\overline{\mathcal{F}}$ the topological closure of a subset $\mathcal{F}$ of $\mathfrak{P}(E)$. Recall that a compact totally disconnected space is called a *Stone space*, whereas a *Priestley space* is a set X together with a topology and an ordering which is compact and *totally order disconnected* in the sense that for every $x, y \in X$ such that $x \not\leq y$ there is some clopen initial segment containing y and not x. Closed subspaces of $\mathfrak{P}(E)$, with the inclusion order added, are Priestley spaces [22]. For an example, we recall that if L is an algebraic lattice then, with the topology induced by the product topology on $\mathcal{J}(K(L))$, it becomes a Priestley space. Priestley spaces are associated to bounded distributive lattices as Stone spaces are associated to Boolean algebras. We will recall in Section 4 the properties we need about the relationship between Priestley spaces and distributive lattices. We refer to [22] and to [7] for an introduction to Stone-Priestley duality and to [13] for more on topologically ordered structures.

1.2. **Basic facts.** We will need the following basic result due to O.Ore and T.Hiraguchi (see [25]):

Lemma 1. *Let P be a poset and κ be a cardinal. The order on P is the intersection of κ linear orders if and only if P is embeddable in a product of κ chains.*

Lemma 2. *(1) Let P and Q be two posets. If P is embeddable in Q then $\mathcal{J}(P)$ is embeddable in $\mathcal{J}(Q)$.*

(2) *Let $(P_i)_{i\in I}$ be a family of posets, then $\mathcal{J}(\Pi_{i\in I}P_i)$ is order-isomorphic to $\Pi_{i\in I}\mathcal{J}(P_i)$ provided that I is finite.*

Proof. The proof of Item 1 is immediate. For Item 2, let A be a subset of $Q := \Pi_{i\in I}P_i$. Given $i \in I$, let $p_i : Q \to P_i$ be the i-th projection and $p_i[A]$ be the image of A. Finally, set $\overline{p}(A) := (p_i[A])_{i\in I}$. We prove that $\overline{p}$ induces an order-isomorphism from $\mathcal{J}(Q)$ onto $\Pi_{i\in I}\mathcal{J}(P_i)$. From this, Item 2 follows. Let $A \in \mathcal{J}(Q)$. First, we claim that $\overline{p}(A) \in \Pi_{i\in I}\mathcal{J}(P_i)$. Indeed, let $i \in I$. Since p_i is order-preserving and A is up-directed, $p_i[A]$ is up-directed. Furthermore, $p_i[A] \in \mathbf{I}(P_i)$. Indeed, let $x \in P_i$ and $y \in p_i[A]$ such that $x \leq y$. Let $\overline{y} \in A$ such that $p_i(\overline{y}) = y$. Let $\overline{x} \in \Pi_{i\in I}P_i$ defined by $\overline{x}_i := x$ and $\overline{x}_j := y_j$ for $j \neq i$. Then $\overline{x} \leq \overline{y}$. Since $A \in \mathbf{I}(\Pi_{i\in I}P_i)$, $\overline{x} \in A$, and thus $x \in p_i[A]$, proving that $p_i[A] \in \mathbf{I}(P_i)$. Since $p_i[A]$ is up-directed, $p_i[A] \in \mathcal{J}(P_i)$. Thus $\overline{p}(A) \in \Pi_{i\in I}\mathcal{J}(P_i)$ as claimed. Next, let $\overline{A} := (A_i)_{i\in I} \in \Pi_{i\in I}\mathcal{J}(P_i)$. Then, trivially, $\Pi_{i\in I}A_i \in \mathcal{J}(Q)$. Since all A_i's are non-empty, $\overline{p}(\Pi_{i\in I}A_i) = \overline{A} \in \mathcal{J}(Q)$, proving that $\overline{p}$ is surjective. To conclude that $\overline{p}$ is an isomorphism, we note that $A = \Pi_{i\in I}p_i[A]$ for every $A \in \mathcal{J}(Q)$. Indeed, we have trivially $A \subseteq \Pi_{i\in I}p_i[A]$. For the reverse inclusion, let $\overline{x} \in \Pi_{i\in I}p_i[A]$. For each $i \in I$, select $\overline{y}(i) \in A$ such that $\overline{y}(i)_i = \overline{x}_i$. Since I is finite and A is up-directed, there is $\overline{z} \in A$ which majorizes each $\overline{y}(i)$. Due to our choices, $\overline{z}$ majorizes $\overline{x}$. Thus $\overline{x} \in A$, as required. □

Let E be a set and $\mathcal{F}$ be a subset of $\mathfrak{P}(E)$. We say that $\mathcal{F}$ is *closed under intersections* if $\cap\mathcal{F}' \in \mathcal{F}$ for every subset $\mathcal{F}'$ of $\mathcal{F}$ (with the convention that $\mathcal{F}' = E$ if $\mathcal{F}' = \emptyset$). We denote by $\mathcal{F}^{\wedge}$ the set of intersections of members of $\mathcal{F}$, (in particular $E \in \mathcal{F}^{\wedge}$. Hence $\mathcal{F}$ is closed under intersections if and only if $\mathcal{F} = \mathcal{F}^{\wedge}$. Sets closed under intersections are usually called *Moore families.* As it is well known, a Moore family $\mathcal{F}$ is topologically closed in $\mathfrak{P}(E)$ if and only if it is closed under unions of up-directed subfamilies. Moore families correspond to closure systems, those which are topology closed to *algebraic closure systems* [14], [13]. We will need the following fact.

Proposition 1. *Let E be a set and $\mathcal{F}$ be a subset of $\mathfrak{P}(E)$. Then $\overline{\mathcal{F}}^{\wedge} = \overline{\mathcal{F}^{\wedge}}$.*

Proof. It relies on the following claims.

Claim 1. *$\overline{\mathcal{F}^{\wedge}}$ is closed under intersections.*

Proof of Claim 1. Set $\mathcal{G} := \overline{\mathcal{F}^{\wedge}}$. Let $\mathcal{G}' \subseteq \mathcal{G}$ and $X := \bigcap\mathcal{G}'$. We prove that $X \in \mathcal{G}$. For that we prove that $O(F,G) \cap \mathcal{F}^{\wedge} \neq \emptyset$ for each finite $F \subseteq X$ and finite $G \subseteq E \setminus X$. We may suppose $X \neq E$ (otherwise, since $E \in \mathcal{F}^{\wedge}$, $X \in \mathcal{G}$ as required). Let $a \in G$. Since $X = \bigcap\mathcal{G}'$ there is some $X_a \in \mathcal{G}'$ such that $X \subseteq X_a \subseteq E \setminus \{a\}$. Since $X_a \in \mathcal{G}$, there is some $Y_{F,a} \in O(F,\{a\}) \cap \mathcal{F}^{\wedge}$. Let $X_F := \cap_{a\in G}X_a$. Clearly $X_F \in O(F,G) \cap \mathcal{F}^{\wedge}$. This proves our claim □

Claim 2. *$\overline{\mathcal{F}}^{\wedge}$ is topologically closed.*

Proof of Claim 2. Set $\mathcal{G} := \overline{\mathcal{F}}^{\wedge}$. Let $X \in \overline{\mathcal{G}}$. Then $O(F,G) \cap \mathcal{G} \neq \emptyset$ for each finite $F \subseteq X$ and finite $G \subseteq E \setminus X$. This implies that for every finite subset $F \subseteq X$, $a \in E \setminus X$, $O(F,\{a\}) \cap \overline{\mathcal{F}} \neq \emptyset$. Let $a \in E \setminus X$. Since $\overline{\mathcal{F}}$ is compact, the intersection $\bigcap\{O(F,\{a\}) : F \subseteq X,\ F \text{ finite}\} \cap \mathcal{F}$ is non empty. Pick X_a in this intersection. Let $X' := \bigcap_{a\in E\setminus X} X_a$. Then $X' \in \overline{\mathcal{F}}^{\wedge} = \mathcal{G}$. But, since each X_a contains X, $X = X'$, hence $X \in \mathcal{G}$. It follows that $\overline{\mathcal{G}} = \mathcal{G}$. This proves our claim. □

From Claim 1 we deduce that $\overline{\mathcal{F}}^{\wedge}$ is included into $\overline{\mathcal{F}^{\wedge}}$ and from Claim 2 the reverse inclusion. □

If P is a poset, we have $N(P) = down(P)^{\wedge}$. Hence, Proposition 1 yields immediately:

Corollary 1. $\overline{N(P)} = \overline{down(P)}^{\wedge}$.

We recall the following fact ([1] Corollary 2.4).

Lemma 3.

$$down(P) \subseteq \mathcal{J}(P) \subseteq \overline{down(P)} \setminus \{\emptyset\}. \tag{2}$$

In particular, the topological closures in $\mathfrak{P}(P)$ of $down(P)$ and $\mathcal{J}(P)$ are the same.

Lemma 4. *Let P be a join-semilattice with a least element. Then:*

$$\overline{down(P)} = \overline{N(P)} = \mathcal{J}(P). \tag{3}$$

Proof. We start with the following:

Claim 3.

$$down(P) \subseteq N(P) \subseteq \mathcal{J}(P). \tag{4}$$

Proof of Claim 3. Trivially, $down(P) \subseteq \mathcal{J}(P)$. Since P is a join-semilattice with a least element, $\mathcal{J}(P)$ is closed under intersection. Hence, $N(P)$ which is made of the intersections of members of $down(P)$ is included into $\mathcal{J}(P)$. □

With Lemma 3 this yields:

$$\overline{down(P)} \subseteq \overline{N(P)} \subseteq \overline{\mathcal{J}(P)} \subseteq \overline{down(P)}. \tag{5}$$

To conclude, we note that $\mathcal{J}(P)$ is topologically closed. Indeed, $\mathcal{J}(P)$ is closed under union of up-directed sets and as, observed above, it is closed under intersection. □

1.3. **Incidence structures and coding.** Let E, F be two sets. A *binary relation from E to F* is any subset ρ of the cartesian product $E \times F$. As usual, we denote by $x\rho y$ the fact that $(x, y) \in \rho$ and by $x\neg\rho y$ the negation. The triple $R := (E, \rho, F)$ is an *incidence structure*; its *complement* is $\neg R := (E, \neg\rho, F)$, where $\neg\rho := E \times F \setminus \rho$, whereas its *dual* is $R^{-1} := (F, \rho^{-1}, E)$, where $\rho^{-1} := \{(y, x) : (x, y) \in \rho\}$. We set $L_R(Y) := \{x \in E : \{x\}\rho Y\}$, resp. $U_R(X) := \{y \in F : X\rho\{y\}\}$, for each $Y \subseteq F$, resp. $X \subseteq E$. And we use $L_R(y)$ and $U_R(x)$ for $L_R(\{y\})$ and $U_R(\{x\})$. With these notations, we have $U_R(X) = L_{R^{-1}}(X)$. The sets $Gal(R) := \{L_R(Y) : Y \subseteq F\}$ and $Gal(R^{-1}) := \{U_R(X) : X \subseteq E\}$ are closed under intersection; hence, once ordered by inclusion, they are complete lattices. Ordered by inclusion, $Gal(R)$ is the *Galois lattice* of R. A fundamental result is that $Gal(R^{-1})$ is isomorphic to $Gal(R)^*$, the *dual* of $Gal(R)$. If $P := (E, \leq)$ is a poset, $Gal((E, \not\geq, E)) = \mathbf{I}(P)$, whereas $Gal((E, \leq, E)) = N(P)$.

Let $R := (E, \rho, F)$, $R' := (E', \rho', F')$ be two incidence structures, a *coding from R into R'* is a pair of maps $f : E \to E'$, $g : F \to F'$ such that

$$x\rho y \Longleftrightarrow f(x)\rho' g(y)$$

for all $x \in E$ and $y \in F$. When such a pair exists, we say that R has a *coding into R'*.

Example 1. *If $R := (E, \rho, F)$ is an incidence structure, the pair (f, g), where $f(x) := L_R \circ U_R(x)$ for $x \in E$ and $g(y) := L_R(y)$ for $y \in F$, is a coding from R into $(Gal(R), \subseteq, Gal(R))$.*

If $E = F$ and $E' = F'$, the pairs (E, ρ), (E', ρ') are *binary relational structures* (or simply, *directed graphs*) and a map $f : E \to E'$ is an *embedding* if it is one-to-one and

$$x \rho y \Longleftrightarrow f(x) \rho' f(y)$$

for all $x, y \in E$. When such a map exists, we say that (E, ρ) *is embeddable into* (E', ρ').

Example 2. *If ρ and ρ' are two orders and (E', ρ') is a complete lattice, R has a coding into R' if and only if (E, ρ) is embeddable in (E, ρ').*

Bouchet's Coding theorem ([4], see also [5]) is a striking illustration of the links between coding and embedding.

Theorem 6. *Let T be a complete lattice and R be an incidence structure, then R has a coding into $(T, \leq, T)$ if and only if $Gal(R)$ is embeddable in T.*

Corollary 2. *Let $R := (E, \rho, F)$ and $R' := (E', \rho', F')$ be two incidence structures. Then $Gal(R)$ is embeddable in $Gal(R')$ whenever R has a coding into R'.*

We will need the following strengthening of Corollary 2.

Theorem 7. *Let $R := (E, \rho, F)$ and $R' := (E', \rho', F')$ be two incidence structures. If R has a coding into R' then there is an embedding ϕ from $\overline{Gal(R)}$ into $\overline{Gal(R')}$ and a continuous and order preserving map ψ from a closed subspace $\mathcal{H}$ of $\overline{Gal(R')}$ onto $\overline{Gal(R)}$ such that $\psi \circ \phi = 1_{\overline{Gal(R)}}$.*

Proof. Let (f, g) be a coding from R into R'. Let $f^d : \mathfrak{P}(E') \to \mathfrak{P}(E)$ be defined by $f^d(X') := f^{-1}(X')$ for $X' \subseteq E'$. The map f^d is continuous. With the fact that $\mathfrak{P}(E')$ is compact, it follows that:

$$f^d[\overline{\mathcal{F}'}] = \overline{f^d[\mathcal{F}']} \tag{6}$$

for every $\mathcal{F}' \subseteq \mathfrak{P}(E')$.

Let $\mathcal{F} := \{L_{R'}(g[Y]) : Y \subseteq F\}$. Clearly, $\mathcal{F}$ is closed under intersection and included into $Gal(R')$. Furthermore, since (f, g) is a coding:

$$f^d(L_{R'}(g(y))) = L_R(y) \tag{7}$$

for every $y \in F$.

Hence,

$$f^d(L_{R'}(g[Y])) = L_R(Y) \tag{8}$$

for every $Y \subseteq F$.

This implies:

$$f^d[\mathcal{F}] = Gal(R). \tag{9}$$

With equation (6), this yields:

$$f^d[\overline{\mathcal{F}}] = \overline{Gal(R)}. \tag{10}$$

Set $\mathcal{H} := \overline{\mathcal{F}}$ and $\psi := f^d_{\restriction \mathcal{H}}$. Clearly $\mathcal{H}$ is a closed subset of $Gal(R')$ and ψ is a continuous and order preserving map from $\mathcal{H}$ into $\overline{Gal(R)}$. Let $X \in \overline{Gal(R)}$. According to equation (10), X belongs to the range of ψ. Set $\phi(X) := \cap\psi^{-1}(X)$. Since $\mathcal{F}$ is closed under intersections, $\overline{\mathcal{F}}$ is closed under intersections too (Claim 1). By definition, ψ preserves intersections. It follows that $\psi(\phi(X)) = X$. From this fact, $\phi(X)$ is the least member X' of $\mathcal{H}$ such that $\psi(X') = X$. This and the fact that ψ preserves intersections imply that ϕ is order preserving. □

Remark 1. *The map ϕ in the proof of Theorem 7 above does not need to be continuous. For an example, take $R := (P, \not\geq, P)$, $R' := (P', \not\geq, P')$ where P and P' are two posets type $1 + \omega^*$ and $(1 \oplus 1) + \omega^*$ respectively (here, $1 \oplus 1$ denotes a 2-element antichain) and, as a coding from R to R', the pair (f, f) where f is an embedding from P into P'.*

From Theorem 7, Lemma 4 and Lemma 2, we derive the following result.

Proposition 2. *If an incidence structure $R := (E, \rho, F)$ has a coding in $(Q, \leq, Q)$, where $Q := \Pi_{i\in I} C_i$ is a finite product of chains, then $\overline{Gal(R)}$ is embeddable in the product $\Pi_{i\in I}\mathbf{I}(C_i)$.*

Proof. Set $C'_i := 1 + C_i$ for each $i \in I$ and $Q' := \Pi_{i\in I} C'_i$. A coding from $(P, \leq, P)$ in $(Q, \leq, Q)$ induces a coding from $(P, \leq, P)$ in $(Q', \leq, Q')$. According to Theorem 7, such a coding yields an embedding from $\overline{Gal(R)}$ into $\overline{N(Q')} = \overline{Gal((Q', \leq, Q'))}$. The poset Q' is a join-semilattice with a least element, hence according to Lemma 4, $\overline{N(Q')} = \mathcal{J}(Q')$. According to Lemma 2, $\mathcal{J}(Q')$ is isomorphic to $\Pi_{i\in I}\mathcal{J}(C'_i)$. To conclude, observe that $\mathcal{J}(C'_i) = \mathbf{I}(C_i)$. □

We need also the following properties:

Lemma 5. *Let (f, g) be a coding from $R := (E, \rho, F)$ into $R' := (E', \rho', F')$ and $(\rho'_i)_{\in I}$ such that $\rho' := \bigcap_{i\in I} \rho'_i$ then $\rho = \bigcap_{i\in I} \rho_i$ where $\rho_i := \{(x, y) \in E \times F : f(x)\rho'_i g(y)\}$.*

The proof is immediate.

Lemma 6. *Let $R := (E, \rho, F)$ be an incidence structure.*

(1) If $\rho = \bigcap_{i\in I} \rho_i$ where each ρ_i is an incidence relation from E to F, then $Gal(R)$ is embeddable in $T := \Pi_{i\in I} Gal(R_i)$ where $R_i := (E, \rho_i, F)$.

(2) If $Gal(R)$ is embeddable in a product $C := \Pi_{i\in I} C_i$ of posets, then $\rho = \bigcap_{i\in I} \rho_i$ where each ρ_i is an incidence relation from E to F such that $Gal((E, \rho_i, F))$ is embeddable in $N(C_i)$.

Proof. (1). Let $A \subseteq E$. Set $\varphi(A) := (L_{R_i} \circ U_{R_i}(A))_{i\in I}$. Clearly:

$$A \subseteq B \text{ implies } \varphi(A) \subseteq \varphi(B). \tag{11}$$

Hence φ is an order-preserving map from $\mathfrak{P}(E)$ into T. In particular, its restriction to $Gal(R)$ is order-preserving. The fact that this is an embedding is an immediate consequence of the following:

Claim 4.

$$A = \bigcap_{i\in I} L_{R_i} \circ U_{R_i}(A) \textit{ provided that } A = L_R \circ U_R(A). \tag{12}$$

Proof of Claim 4. From $\rho \subseteq \rho_i$ for all $i \in I$, we have $A \subseteq \bigcap_{i\in I} L_{R_i} \circ U_{R_i}(A) \subseteq \bigcap_{i\in I} L_{R_i} \circ U_R(A)$. From $\rho = \bigcap_{i\in I} \rho_i$ we get $\bigcap_{i\in I} L_{R_i}(B) = L_R(B)$ for every $B \subseteq F$. Applying this to $B := U_R(A)$, we get $A \subseteq \bigcap_{i\in I} L_{R_i} \circ U_{R_i}(A) \subseteq \bigcap_{i\in I} L_{R_i} \circ U_R(A) = L_R \circ U_R(A)$. The claim follows immediately. □

(2). Let $c : Gal(R) \to C$ be an embedding and $p_i : C \to C_i$ be the $i-th$-projection. Set $f(x) := L_R \circ U_R(x)$ for $x \in E$ and $g(y) := L_R(y)$ for $y \in F$. Set $f_i := p_i \circ c \circ f$, $g_i := p_i \circ c \circ g$ and $\rho_i := \{(x, y) \in E \times F : f_i(x) \leq_i g_i(x)\}$. Then (f_i, g_i) is a coding from $R_i := (E, \rho_i, F)$ into $(C_i, \leq_i, C_i)$. Thus, from Lemma 2, $Gal(R_i)$ is embeddable in $Gal((C_i, \leq_i, C_i)) = N(C_i)$. To conclude observe that (f, g) is a coding from R into $(Gal(R), \subseteq, Gal(R))$, hence $\rho = \bigcap_{i\in I} \rho_i$. □

1.4. **Ferrers relations, interval orders and dimensions.** Let $R := (E, \rho, F)$ be an incidence structure. The binary relation ρ from E to F is a *Ferrers relation* if for every $x, x' \in E$, $y, y' \in F$, $x\rho y$ and $x'\rho y'$ imply $x\rho y'$ or $x'\rho y$. As it is well known, ρ is Ferrers if and only if $Gal(R)$ is a chain. It follows from Bouchet's theorem that *$Gal(R)$ is a chain if and only if R has a coding into $(C, \leq, C)$ where C is a chain.*

Let C be a chain, an *interval* of C is any subset I of C such that $x, y \in I, z \in C$ and $x < z < y$ imply $z \in I$. One may order the set $Int(C)$ of non empty intervals of C by setting $I < J$ if $x < y$ for all $x \in I$ and $y \in J$. Let P be a poset; the order on P is an *interval order*, and by extension P too, if P is isomorphic to a subset of $Int(C)$ for some chain C. We recall that:

Lemma 7. *A poset P is an interval order if and only if $(P, <, P)$ is a Ferrers relation, or equivalently $(P, <, P)$ has a coding into a chain.*

Let $\mathcal{F}$, resp. $\mathcal{J}$, be the class of Ferrers relations, resp. interval orders. We recall that the *Ferrers dimension* of an incidence structure $R := (E, \rho, F)$ is the least cardinal κ such that ρ is the intersection of κ *Ferrers relations* from E to F. We denote it by $\mathcal{F} - dim(R)$. The *interval dimension* of P is the smallest cardinal κ such that the order on P is the intersection of κ interval orders. We denote it by $\mathcal{I} - dim(P)$. We recall two basic results relating theses notions, due to Bouchet [5] and Cogis [6], namely:

$$\mathcal{F} - dim((P, \leq, P)) = dim(P) \tag{13}$$

and

$$\mathcal{F} - dim((P, <, P)) = \mathcal{I} - dim(P) \tag{14}$$

for every poset P.

These three notions of dimension: order dimension, Ferrers dimension and interval dimension are based on three classes of structures: chains, Ferrers relations and interval orders and are expressible in terms of Galois lattices. Replacing theses these classes by others yield other notions of dimension that we discuss at the end of this section.

1.5. **Bipartite posets.** A poset is *bipartite* if this is the union of two antichains. We recall the following result:

Lemma 8. *Let Q be a bipartite poset. Then*

$$\mathcal{I} - dim(Q) \leq dim(Q) \leq \mathcal{I} - dim(Q) + 1. \tag{15}$$

Let $R := (E, \rho, F)$ be an incidence structure. The *bipartite poset associated to* R, denoted by $B(R)$, is the poset whose base set is $E' := E \times \{0\} \cup F \times \{1\}$ ordered by:

$$(x, i) < (y, j) \text{ if } (x, y) \in \rho \text{ and } i < j.$$

If $P := (E, \leq)$ we set $B(P) := B(E, \leq, E)$ and $B(\check{P}) := B(E, <, E)$. The posets $B(P)$ and $B(\check{P})$ are respectively called the *split* and the *open split* of P.

We note that if R is an incidence structure then R has a coding into $(B(R), \leq, B(R))$ as well as in $(B(R), <, B(R))$. In particular:

(16) $$Gal(R) \text{ is embeddable into } N(B(R)).$$

As a corollary of (16) it turns out that for every poset P:

(17) $$N(P) \text{ is embeddable into } N(B(P)).$$

Note also that:

Lemma 9. *If P is a poset, $B(P)$ is embeddable in a product $P \times C$ where C is a chain of the form $D + D$, the order type of D being given by any linear extension of P^*.*

We will use the following easy fact:

Lemma 10. *Let $R := (E, \rho, F)$ and $R' := (E', \rho, F')$ be two incidence structures. Every coding (f, g) from R to R' such that f and g are one to one induces an embedding of $B(R)$ in $B(R')$. The converse holds if for every $x \in E$ there is some $y \in F$ such that $(x, y) \in \rho$.*

We also recall the following result of Bouchet and Cogis:

(18) $$\mathcal{F} - dim(R) = \mathcal{I} - dim(B(R)) = dim(Gal(R)).$$

The first equality in (18) added to equality (13) yields:

(19) $$dim(P) = \mathcal{I} - dim(B(P)).$$

Similarly, the first equality in (18) added to equality (14) yields:

(20) $$\mathcal{I} - dim(P) = \mathcal{I} - dim(B(\check{P})).$$

Inequalities (15) with equality (19) yield

(21) $$dim(P) \leq dim(B(P)) \leq dim(P) + 1.$$

Similarly, inequalities (15) with equality (20) yield

(22) $$\mathcal{I} - dim(P) \leq dim(B(\check{P})) \leq \mathcal{I} - dim(P) + 1.$$

Inequalities (21) are due to Kimble (cf. [27]).

Let $\underline{2}.P$ the ordinal product of the two-element chain $\underline{2}$ by a poset P. This is the set of pairs (x, i), with $x \in P$, $i \in 2$, lexicographically ordered (that is $(x, i) \leq (x', i')$ if either $x < x'$ or $x = x'$ and $i < i'$).

Lemma 11. *Let P be a poset and $Q := \underline{2}.P$ then:*

(1) $B(P)$ is embeddable in $B(\check{Q})$.
(2) $B(\check{P})$ is embeddable in $B(Q)$.

Proof. Item (1). Let f and g be the maps from P to Q defined by $f(x) := (x, 0)$ and $g(x) := (x, 1)$. Then (f, g) is a one-to one coding of $(P, \leq, P)$ in $(Q, <, Q)$. This coding induces an embedding from $B(P)$ in $B(\check{Q})$.

Item(2). Let f' and g' be the maps from P to Q defined by $f'(x) := g(x)$ and $g'(x) := f(x)$. Then (f', g') is a one-to one coding of $(P, <, P)$ in $(Q, \leq, Q)$. This coding induces an embedding from $B(\check{P})$ in $B(Q)$. □

Proposition 3. *$B(P)$ and $B(\check{P})$ are embeddable in each other whenever $\underline{2}.P$ is embeddable in P.*

Proof. Let $Q := \underline{2}.P$. Suppose that Q is embeddable in P. Then $B(Q)$ is embeddable in $B(P)$. According to item (2) of Lemma 11, $B(\check{P})$ is embeddable in $B(Q)$. Hence $B(\check{P})$ is embeddable in $B(P)$. Similarly, $B(\check{Q})$ is embeddable in $B(\check{P})$. According to item (1) of Lemma 11, $B(P)$ is embeddable in $B(\check{Q})$. Hence $B(P)$ is embeddable in $B(\check{P})$. □

1.6. **A relativisation of the notions of dimension.** Let $\mathcal{R}$ be a class of incidence structures and let $R := (E, \rho, F)$ be an incidence structure. If ρ is the intersection of incidence relations ρ_i such that $(E, \rho_i, F) \in \mathcal{R}$, we define the *$\mathcal{R}$-dimension* of R, that we denote by $\mathcal{R} - dim(R)$, as the least cardinal κ such that ρ is the intersection of κ such relations. Let $\mathcal{D}$ be a class of posets and let P be a poset. If the order $\leq$ is the intersection of orders $\leq_i$ such that $(E, \leq_i) \in \mathcal{D}$, the *$\mathcal{D}$-dimension* of P, that we denote by $\mathcal{D} - dim(P)$, is the least cardinal κ such that $\leq$ is the intersection of κ such orders. If the poset P is embeddable in a product of members of $\mathcal{D}$ we denote by $\mathcal{D} - \pi dim(P)$ the least cardinal κ such that P is embeddable in a product of κ members of $\mathcal{D}$. For example, if $\mathcal{R}$ is the class $\mathcal{F}$ of Ferrers relations, $\mathcal{R} - dim(R)$ is the Ferrers dimension of R. If $\mathcal{D}$ is the class $\mathcal{L}$ of chains, $\mathcal{D} - dim(P)$ is the order dimension of P and if $\mathcal{D}$ is the class of interval orders, $\mathcal{D} - dim(P)$ is the interval dimension of P.

Definition 12. *A class $\mathcal{C}$ of posets is* dimensional *if:*

(1) $\underline{2} \in \mathcal{C}$.
(2) If $C \in \mathcal{C}$ and C' is embeddable in C then $C' \in \mathcal{C}$.
(3) If $C \in \mathcal{C}$ then $N(C) \in \mathcal{C}$.

Let $(C_i)_{i \in I}$ be a family of posets such that I is equipped with a well-ordering. The *lexicographical product* of this family is the poset denoted $\bigodot_{i \in I} C_i$ whose underlying set is the cartesian product $\Pi_{i \in I} C_i$, the ordering being defined by:

$$(x_i)_{i \in I} \leq (y_i)_{i \in I}$$

if either $(x_i)_{i \in I} = (y_i)_{i \in I}$ or $x_{i_0} < y_{i_0}$ where i_0 is the least $i \in I$ such that $x_{i_0} \neq y_{i_0}$.

Proposition 4. *Let $\mathcal{C}$ be a dimensional class of posets and $Gal^{-1}(\mathcal{C})$ be the class of incidence structures S such that $Gal(S) \in \mathcal{C}$. Then:*

(i) $Gal^{-1}(\mathcal{C}) - dim(R) = \mathcal{C} - \pi dim(Gal(R))$ for every incidence structure $R := (E, \rho, F)$.
(ii) $Gal^{-1}(\mathcal{C}) - dim((P, \leq, P)) = \mathcal{C} - \pi dim(P)$ for every poset P.
(iii) If $\mathcal{I}(\mathcal{C})$ is the class of posets $(L, \leq)$ such that $Gal((L, <, L)) \in \mathcal{C}$ then $\mathcal{C} - \pi dim(Gal((P, <, P))) \leq \mathcal{I}(\mathcal{C}) - dim(P) \leq \mathcal{C} - dim(Gal((P, <, P)))$.

Let κ be a cardinal. If $\bigodot_{i \in I} C_i \in \mathcal{C}$ whenever $(C_i)_{i \in I}$ is a family of members of $\mathcal{C}$ such that $|I| < \kappa$ then:

(i') $\mathcal{C}-dim(P)=\mathcal{C}-\pi dim(P)$ for every poset P such that $\mathcal{C}-\pi dim(P)<\kappa$.
(ii') $\mathcal{I}(\mathcal{C})-dim(P)=\mathcal{C}-dim(Gal((P,<,P)))$ for every poset P such that $\mathcal{C}-\pi dim(Gal((P,<,P)))<\kappa$.

Proof. Observe that since a poset P is embeddable in the power set $\mathfrak{P}(P)$ ordered by inclusion, and since this poset is isomorphic to the power $\underline{2}^P$, P is embeddable in a power of $\underline{2}$. Since $\underline{2}\in\mathcal{C}$, $\mathcal{C}-\pi dim(P)$ is well-defined.
Item (i). Let $\kappa:=\mathcal{C}-\pi dim(Gal(R))$. According to the observation above, this quantity is well-defined. Let $C:=\Pi_{i\in I}C_i$ be a product of κ members of $\mathcal{C}$ such that $Gal(R)$ is embeddable in C. According to Item (2) of Lemma 6, $\rho=\bigcap_{i\in I}\rho_i$ where each ρ_i is an incidence relation from E to F such that $Gal((E,\rho_i,F))$ is embeddable in $N(C_i)$. Since $\mathcal{C}$ is dimensional, $Gal((E,\rho_i,F))\in\mathcal{C}$, hence $Gal^{-1}(\mathcal{C})-dim(R)$ is well-defined and $Gal^{-1}(\mathcal{C})-dim(R)\leq\mathcal{C}-\pi dim(Gal(R))$. The converse inequality follows immediately from Item (1) of Lemma 6.
Item (ii). We have $N(P)=Gal((P,\leq,P))$. Hence, from Item (i), we have $Gal^{-1}(\mathcal{C})-dim((P,\leq,P))=\mathcal{C}-\pi dim(N(P))$. Since P is embeddable in $N(P)$, $\mathcal{C}-\pi dim(P)\leq\mathcal{C}-\pi dim(N(P))$. To get the converse inequality, note that if P is embeddable in a product $C:=\Pi_{i\in I}C_i$ then, since each C_i is embeddable in $N(C_i)$, C is embeddable in $C':=\Pi_{i\in I}N(C_i)$, hence P is embeddable in C'. Since C' is a complete lattice, $N(P)$ is embeddable in C'. From the fact that $\mathcal{C}$ is dimensional, $C'\in\mathcal{C}$. The result follows.
Item (iii). Set $R:=(P,<,P)$. We prove first the second inequality.

Claim 5. *$\mathcal{I}(\mathcal{C})-dim(P)\leq\mathcal{C}-dim(Gal(R))$.*

Proof of Claim 5. Let (f,g) be the coding from R into $(Gal(R),\subseteq,Gal(R))$ defined by $f(x):=L_R\circ U_R(x)$ for $x\in P$ and $g(y):=L_R(y)$ for $y\in P$. We have:

$$g(x)\subset f(x) \tag{23}$$

for all $x\in P$. Indeed, since $x\not<x$, we have $f(x)\not\subseteq g(x)$; on an other hand we have $g(x)=L_R(x)\subseteq L_R\circ U_R(x)=f(x)$. Now, let $\mathcal{L}'$ be an order extending the inclusion order on $Gal(R)$. Set $\mathcal{L}:=\{(x,y)\in P:(f(x),g(y))\in\mathcal{L}'\}$. Then $\mathcal{L}$ is irreflexive and transitive. Indeed, according to (23) we have $g(x)\subset f(x)$,thus $(g(x),f(x))\in\mathcal{L}'$. This implies that $(f(x),g(x))\notin\mathcal{L}$, hence $(x,x)\notin\mathcal{L}$, proving that $\mathcal{L}$ is irreflexive. Let $(x,y),(y,z)\in\mathcal{L}$. Since $g(y)\subset f(y)$, $(g(y),f(y))\in\mathcal{L}'$. This easily yields that $(x,y)\in\mathcal{L}$, thus $\mathcal{L}$ is transitive. If, moreover $(Gal(R),\mathcal{L}')\in\mathcal{C}$, $Gal((P,\mathcal{L},P))\in\mathcal{C}$. Indeed, (f,g) is a coding from $(P,\mathcal{L},P)$ into $(Gal(R),\mathcal{L}',Gal(R))$. Hence, from Bouchet's theorem (cf. Corollary 2), $Gal((P,\mathcal{L},P))$ is embeddable into $Gal((Gal(R),\mathcal{L}',Gal(R)))=N((Gal(R),\mathcal{L}'))$. Since $\mathcal{C}$ is dimensional, if $(Gal(R),\mathcal{L}')\in\mathcal{C}$, $N((Gal(R),\mathcal{L}'))\in\mathcal{C}$ too. Thus $Gal((P,\mathcal{L},P))\in\mathcal{C}$. With that, our claim follows from Lemma 5. □

Claim 6. *$Gal^{-1}(\mathcal{C})-dim(R)\leq\mathcal{I}(\mathcal{C})-dim(P)$.*

Proof of Claim 6. Trivial. □.

From (i) we have $Gal^{-1}(\mathcal{C})-dim(R)=\mathcal{C}-\pi dim(Gal(R))$. Thus, with Claim 6, $\mathcal{C}-\pi dim(Gal(R))\leq\mathcal{I}(\mathcal{C})-dim(P)$. This is the first inequality. With that, the proof of Item (iii) is complete.
Item (i'). We have $\mathcal{C}-\pi dim(P)\leq\mathcal{C}-dim(P)$ without any condition on $\mathcal{C}$. Indeed, if the order $\leq$ on P is the intersection of a family $(\leq_i)_{i\in I}$ of orders on P, the map $\delta:P\to P^I$ defined by $\delta(x)(i):=x$ is an embedding of P in the direct product

$\Pi_{i\in I}P_i$ where $P_i := (P, \leq_i)$. Conversely, suppose that there is an embedding from P in a direct product $Q := \Pi_{i\in I}P_i$, with $P_i \in \mathcal{C}$. Let P' be the image of P.

Claim 7. *The order on Q is the intersection of $|I|$ orders $\leq_i$ such that $(Q, \leq_i) \in \mathcal{C}$.*

Proof of Claim 7. For each $i \in I$, choose a well-ordering $\mathcal{L}_i$ on I for which i is the first element and let Q_i be the lexicographical product of the P_i's indexed by $L_i := (I, \mathcal{L}_i)$. The order on Q is the intersection of the orders of the Q_i's. If each P_i belong to $\mathcal{C}$, then with our hypothese on $\mathcal{C}$, the Q_i's belong to $\mathcal{C}$. □

Now, the order on P' is the intersection of the orders induced on P' by the Q_i's. Since $\mathcal{C}$ is dimensional, these orders belong to $\mathcal{C}$, hence $\mathcal{C}-dim(P) \leq |I|$. Thus Item (i') holds.

Item (ii'). Apply Item (i') to $Gal((P, <, P))$ and use Item (iii).

With this, the proof of Proposition 4 is complete. □

Since the class of chains is preserved under lexicographical products, Proposition 4 applied to $\mathcal{C} := \mathcal{L}$ yields formulas (13) and (14).

2. Scattered posets and scattered topological spaces

A poset P, or its order as well, is *scattered* if it does not contain a subset ordered as the chain η of rational numbers; in other words, the chain η is not embeddable in P. A topological space is *scattered* if every non-empty subset has at least an isolated point(w.r.t. the induced topology). Sometimes, to avoid confusion, we use the terms *ordered scattered* and *topologically scattered*. These two notions are quite related. This is particularly the case when the order and the topology are defined on the same universe. For an example, if the ordering is linear and the topology is the interval-topology, the chain is complete if and only if the space compact (Hausdorff). Moreover, *if C is a complete chain, the conditions that C is order-scattered, C is topologically scattered, C is order isomorphic to $\mathbf{I}(D)$, where D is a scattered chain, are equivalent.* From this fact follows that *a chain D is order scattered if and only if its MacNeille completion $N(D)$ is order scattered.*

The class $\mathcal{S}$ of scattered posets is closed downward, that is if $P \in \mathcal{S}$ and Q is embeddable in P then $Q \in \mathcal{S}$. Furthermore, it is closed under finite direct product and under finite lexicographical product. In particular, the class $\mathcal{L}_{\mathcal{S}}$ of scattered chains is preserved under finite lexicographical product. This property, and (i') of Proposition 4, yield an important known fact:

Proposition 5. *Let n be a positive integer. An ordered set P is embeddable in a product of n scattered chains if and only if the order on P is the intersection of n scattered linear orders.*

2.1. Scattered dimensions.

Let $\mathcal{F}_{\mathcal{S}}$, resp. $\mathcal{I}_{\mathcal{S}}$, be the class of incidence structure R, resp. posets P, such that the Galois lattice $Gal(R)$, resp. $Gal((P, <, P))$ belongs to $\mathcal{L}_{\mathcal{S}}$.

The following lemma completes the analogy between $\mathcal{I}_{\mathcal{S}}$ and $\mathcal{I}$

Lemma 13. *A poset P belongs to $\mathcal{I}_{\mathcal{S}}$ if and only if P is isomorphic to a subset of $Int(C)$ for some scattered chain C.*

Let n be an integer, we denote by $\mathcal{F}(n)$, resp. $\mathcal{I}(n)$, resp. $\mathcal{L}(n)$ the class of incidence structures R, resp. of posets P such that $\mathcal{F} - dim(R) \leq n$, resp. $\mathcal{I} - dim(P) \leq n$, resp. $dim(P) \leq n$. We define $\mathcal{F}_{\mathcal{S}}(n)$, resp. $\mathcal{I}_{\mathcal{S}}(n)$, resp. $\mathcal{L}_{\mathcal{S}}(n)$, accordingly.

Theorem 8. *Let n be an integer and let R be an incidence structure, resp. a poset P. Then $R \in \mathcal{F}_{\mathcal{S}}(n)$, resp. $P \in \mathcal{I}_{\mathcal{S}}(n)$, resp. $P \in \mathcal{L}_{\mathcal{S}}(n)$, if and only if $Gal(R)$, resp. $Gal((P,<,P))$, resp. $N(P)$, belongs to $\mathcal{L}_{\mathcal{S}}(n)$.*

Proof. We apply Proposition 4 with $\mathcal{C} := \mathcal{L}_{\mathcal{S}}$. Since $\mathcal{F}_{\mathcal{S}} = Gal^{-1}(\mathcal{L}_{\mathcal{S}})$, Item (i) yields $\mathcal{F}_{\mathcal{S}} - dim(R) = \mathcal{L}_{\mathcal{S}} - \pi dim(Gal(R))$ for every incidence structure $R := (E, \rho, F)$. From Proposition 5, $\mathcal{L}_{\mathcal{S}} - dim(Gal(R)) = \mathcal{L}_{\mathcal{S}} - \pi dim(Gal(R))$ provided that $\mathcal{L}_{\mathcal{S}} - \pi dim(Gal(R)) < \omega$. Thus if $R \in \mathcal{F}_{\mathcal{S}}(n)$, $\mathcal{L}_{\mathcal{S}} - dim(Gal(R)) \leq n$, that is $Gal(R) \in \mathcal{L}_{\mathcal{S}}(n)$. The converse follows from the fact that $\mathcal{L}_{\mathcal{S}} - \pi dim(Gal(R)) \leq \mathcal{L}_{\mathcal{S}} - dim(Gal(R))$. Set $R := (P,<,P)$. Since $\mathcal{I}_{\mathcal{S}} = \mathcal{I}(\mathcal{C})$, Item (iii) yields $\mathcal{L}_{\mathcal{S}} - \pi dim(Gal(R) \leq \mathcal{I}_{\mathcal{S}} - dim(P) \leq \mathcal{L}_{\mathcal{S}} - dim(Gal(R)$. Thus, if $P \in \mathcal{I}_{\mathcal{S}}(n)$, $\mathcal{L}_{\mathcal{S}} - \pi dim(Gal(R) \leq n$. Since $\mathcal{L}_{\mathcal{S}} - \pi dim(Gal(R) = \mathcal{L}_{\mathcal{S}} - dim(Gal(R)$, $Gal(R) \in \mathcal{L}_{\mathcal{S}}(n)$. Again, the converse follows from the fact that $\mathcal{L}_{\mathcal{S}} - \pi dim(Gal(R)) \leq \mathcal{L}_{\mathcal{S}} - dim(Gal(R))$. Now, set $R := (P, \leq, P)$. Combining Item (i) and Item (ii), we get $\mathcal{C} - \pi dim(Gal(R)) = Gal^{-1}(\mathcal{C}) - dim(R) = \mathcal{C} - \pi dim(P)$, hence $\mathcal{C} - \pi dim(N(P)) = \mathcal{C} - \pi dim(P)$. Since $\mathcal{C} - \pi dim(P) \leq \mathcal{C} - dim(P)$, if $P \in \mathcal{L}_{\mathcal{S}}(n)$, $\mathcal{C} - \pi dim(N(P)) \leq n$. With Item (i') we get $N(P) \in \mathcal{L}_{\mathcal{S}}(n)$. The converse is similar. $\square$

With these notations, one may ask:

Questions 9. *Let P be a poset and R be an incidence structure.*

(i) If $\mathcal{L}_{\mathcal{S}} - dim(P)$ is finite does $\mathcal{L}_{\mathcal{S}} - dim(P) = dim(P)$?
(ii) If $\mathcal{I}_{\mathcal{S}} - dim(P)$ is finite does $\mathcal{I}_{\mathcal{S}} - dim(P) = \mathcal{I} - dim(P)$?
(iii) If $\mathcal{F}_{\mathcal{S}} - dim(R)$ is finite does $\mathcal{F}_{\mathcal{S}} - dim(R) = \mathcal{F} - dim(R)$?

Question (i) is just a reformulation of Question 1.

With the help of Theorem 8, one can show that a positive answer to (i) is equivalent to a positive answer to (iii) and implies a positive answer to (ii).

2.2. **Topologically scattered spaces and Galois lattices.** .

Lemma 14. *(1) The continuous image of a compact scattered space is scattered.*
(2) A finite product of scattered topological spaces is scattered.
(3) A Priestley space which is topologically scattered is order scattered.

The first fact is non-trivial, it is due to W.Rudin. The second and third fact are easy and well-known.

Remark If L is a topologically scattered algebraic lattice, the algebraic lattice $\overline{N(L)}$ is not necessarily topologically scattered. A topologically scattered algebraic lattice L containing an infinite independent set X will do. Indeed, recall that a subset X of a lattice L is *independent* if $x \not\leq \vee X$ for every $x \in X$, $F \in [X \setminus \{x\}]^{<\omega}$. Furthermore, if X is independent, $\mathfrak{P}(X)$ is embeddable in $\mathcal{J}(L)$. Thus, if X is infinite, $\mathcal{J}(L)$ is not order scattered. Since $\overline{N(L)} = \mathcal{J}(L)$, this set is not topologically scattered. For that, let P be a countable well-founded poset with no infinite antichain. Set $L = \mathbf{I}(P)$. Then L is countable, thus topologically scattered. Since L is distributive, antichains of join-irreducible members of L are independent subsets of L. To get L containing an infinite antichain of join-irreducibles, take for P the poset made of $\{(m,n) \in \mathbb{N}^2 : m < n\}$ ordered by setting

$$(m,n) \leq_P (m',n') \text{ if either } m = m' \text{ and } n \leq n' \text{ or } n < m' \tag{24}$$

This poset was discovered by R. Rado [23].

Lemma 15. *Let $R := (E, \rho, F)$ and $R' := (E', \rho', F')$ be two incidence structures. If R has a coding into R' and $\overline{Gal(R')}$ is topologically scattered then $\overline{Gal(R)}$ is topologically scattered.*

Proof. According to Theorem 7, $\overline{Gal(R)}$ is the continuous image of a closed subspace of $\overline{Gal(R')}$. From Rudin's result ((1) of Lemma 14) it is topologically scattered. □

Lemma 16. *Let n be an integer and $R := (E, \rho, F)$ be an incidence structure. If $Gal(R)$ is embeddable into a product of n scattered chains, then $\overline{Gal(R)}$ too. Moreover, $\overline{Gal(R)}$ is topologically scattered.*

Proof. Suppose that $Gal(R)$ is embeddable in $Q := \Pi_{i \in I} C_i$ with $|I| = n$. According to Bouchet's theorem (Theorem 6), R has a coding into Q. According to Theorem 7, $\overline{Gal(R)}$ is embeddable in $\overline{Gal((Q, \leq, Q))} = \overline{N(Q)}$. Since $Gal(R)$ has a least element, we may suppose w.l.o.g that Q has a least element, that is each C_i has a least element. Then Q is a join-semilattice with a least element, hence from Lemma 4, $\overline{N(Q)} = \mathcal{J}(Q)$. Since I is finite, Lemma 2 ensures that $\mathcal{J}(Q)$ is order isomorphic to $\Pi_{i \in I} \mathcal{J}(C_i)$, that is to $\Pi_{i \in I} \mathbf{I}(C'_i)$, where each C'_i is such that $C_i = 1 + C'_i$. Since the C'_i's are order scattered, the $\mathbf{I}(C'_i)$'s are order scattered. Their product, being finite, is scattered too. This proves the first assertion. The $\mathbf{I}(C'_i)$'s are in fact topologically scattered. Hence, as a finite product of scattered spaces, $\mathcal{J}(Q)$ is topologically scattered. The second part of the assertion follows from Lemma 15. □

3. Proofs of Theorems 1, 2 and 3

3.1. **Proof of Theorem 3.** $(i) \Rightarrow (ii)$. Suppose that (i) holds. Then, according to Proposition 5, P is embeddable in a product $Q := \Pi_{i \in I} C_i$ of n scattered chains. In particular $(P \leq P)$ has a coding into $(Q, \leq Q)$. According to Proposition 2, $\overline{N(P)}$ is embeddable in $\Pi_{i \in I} \mathbf{I}(C_i)$. Thus (ii) holds. Moreover, from Lemma 16, $\overline{N(P)}$ is topologically scattered. $(ii) \Rightarrow (i)$. Suppose that (ii) holds. Since P is embeddable in $\overline{N(P)}$, it is embeddable in a product of n scattered chains. According to Proposition 5, (i) holds.

3.2. **Proof of Theorem 2.** Suppose that $B(P) \in \mathcal{L}_{\mathcal{S}}(n)$. From Theorem 3, $N(B(P)) \in \mathcal{L}_{\mathcal{S}}(n)$. Since from (17), $N(P)$ is embeddable into $N(B(P))$, $N(P) \in \mathcal{L}_{\mathcal{S}}(n)$. Hence, $P \in \mathcal{L}_{\mathcal{S}}(n)$. Conversely, suppose that $P \in \mathcal{L}_{\mathcal{S}}(n)$. In this case, we apply Lemma 9 with $D \in \mathcal{L}_{\mathcal{S}}$. It turns out that $B(P) \in \mathcal{L}_{\mathcal{S}}(n+1)$. □

3.3. **Proof of Theorem 1.** We prove that if P is one of the ten posets listed in Theorem 1, either $\overline{N(P)}$ or $\overline{N(P^*)}$ is not topologically scattered. According to Theorem 3 the order on P cannot be the intersection of finitely many scattered linear orders and thus any poset containing a copy of P has the same property.

Since for each member P of our list, P^* belongs to our list, it suffices to check that $\overline{N(P)}$ is not topologically scattered in the following cases.

Case 1. $P \in \{\eta, T_2, \Omega(\eta)\}$. If $P = \eta$, $\overline{N(P)} = \mathbf{I}(\eta)$. Topologically, this space is the Cantor set; it is not topologically scattered. If $P = T_2$, then $\overline{N(P)}$ is made of the binary tree plus the maximal branches of the binary tree and a top element added. These maximal branches form a Cantor space, hence $\overline{N(P)}$ is not topologically scattered (a strengthening of this fact will be given in Proposition 7). If $P =$

$\Omega(\eta)$, the pictorial representation of $\Omega(\eta)$ given in Figure 1 show that $\Omega(\eta)$ is a 2-dimensional poset, in fact the intersection of a linear order of type ω and of a linear order of type $\omega.\eta$. Moreover, as it is easy to see, $\mathbf{I}(\eta)$ is embeddable in $\mathcal{J}(\Omega(\eta))$. Since $\overline{\mathcal{J}(\Omega(\eta))} \subseteq \overline{down(\Omega(\eta))} \subseteq \overline{N(\mathcal{J}(\Omega(\eta))}$, it follows that $\overline{N(\mathcal{J}(\Omega(\eta))}$ is not order scattered, hence not topologically scattered.

Case 2. $P := B(\check{Q})$ where $Q \in \{\eta, T_2, \Omega(\eta)\}$. We deal with the three cases at once. Since $N(Q) \setminus \overline{Gal((Q, <, Q))}$ is made of isolated points, it follows from Case 1 that $\overline{Gal((Q, <, Q))}$ is not topologically scattered. Since $(Q, <, Q)$ has a coding into $B((Q, <, Q)) = B(\check{Q})$, Theorem 7 yields that $\overline{Gal((Q, <, Q))}$ is the continuous image of $\overline{N(B(\check{Q}))}$. From Rudin's result ((1) of Lemma 14) this latter set cannot be topologically scattered. □

Lemma 17. *If $P \in \{\eta, T_2, \Omega(\eta)\}$, $B(P)$ and $B(\check{P})$ are embeddable in each other.*

Proof. Observe that $\underline{2}.P$ is embeddable in P and apply Proposition 3. □

Lemma 18. *The ten posets listed in Theorem 1 have dimension at most* 3.

Proof. Trivially η has dimension 1. As a tree, T_2 has dimension 2. Figure 1 shows that $\Omega(\eta)$ has dimension 2. The poset $B(\check{\eta})$ is defined as the strict product of the chain of rational numbers and the 2-element chain on $\{0, 1\}$ with $0 < 1$. Hence, this is a 2-dimensional poset. Let $P \in \{T_2, \Omega(\eta)\}$. Since P has dimension 2, it follows from equation (21) that $B(P)$ has dimension at most 3. According to Lemma 17, $B(\check{P})$ is embeddable in $B(P)$, thus $B(\check{P})$ has dimension at most 3. Let $A := \{0\} \cup 3 \times 2$ ordered so that 0 is the least element and $(i, j) < (i', j')$ if $i = i'$ and $j < j'$. This poset is a tree obtained by taking the direct sum of three copies of a 2-element chain and adding a least element. This tree is obviously embeddable in T_2. Every 2-dimensional poset is embeddable in $\Omega(\eta)$ thus A is also embeddable in $\Omega(\eta)$. Let $X := \{(0, 0)\} \cup \{((i, j), j) : i < 3, j < 2\}$ and $B := B(\check{A})_{\restriction X}$. The poset B is 3-dimensional poset (in fact a 3-irreducible poset). Since A is embeddable in P, $B(\check{A})$ is embeddable in $B(\check{P})$. Thus, $B(\check{P})$ has dimension 3. With the fact that a poset and its dual have the same dimension, our proof is complete. □

Lemma 19. *The ten posets listed in Theorem 1 are pairwise incomparable with respect to embeddability.*

Proof. Let $X_0 := \eta$, $X_1 := T_2$, $X_2 := \Omega(\eta)$, $X_3 := B(\check{\eta})$, $X_4 := B(\check{T}_2)$, $X_5 := B(\check{\Omega}(\eta))$, $X_6 := (T_2)^*$, $X_7 := (\Omega(\eta))^*$, $X_8 := (B(\check{T}_2))^*$, $X_9 := (B(\check{\Omega}(\eta)))^*$. We need to prove that X_i is not embeddable in X_j for all pairs (i, j) of distinct elements. Clearly, it suffices to consider the pairs for which $i \leq 5$ and $j \leq 9$. We consider only pairs (i, j) for which a significant argument is needed. For the pair $(1, 2)$ note that $\mathcal{J}^{\neg\downarrow}(X_1)$ is an antichain whereas $\mathcal{J}^{\neg\downarrow}(X_2)$ is a chain. For pairs $(3, j)$, with $j \notin \{0, 6, 8, 9\}$, note that $\mathbf{I}(X_3)$ contains principal initial segments which are infinite whereas $\mathbf{I}(X_j)$ contains none. For pairs (i, j) such that $i \in \{4, 5\}$ and $j \notin \{4, 5, 8, 9\}$ note that $dim(X_i) = 3$ and $dim(X_j) \leq 2$ (Lemma 18). For pairs (i, j) such that $i \in \{4, 5\}$, $j \in \{3, 4, 5, 8\}$, then we may write $X_i = B(\check{Y}_i)$ and $X_j = B(\check{Y}_j)$. If X_i is embeddable in X_j, it follows from Lemma 17 that $B(Y_i)$ is embeddable in $B(Y_j)$. From Lemma 10 there is a coding from $(Y_i, \leq, Y_i)$ in $(Y_j, \leq, Y_j)$, from which follows that $\overline{N(Y_i)}$ is embeddable in $\overline{N(Y_j)}$. Since $Y_i = X_{i'}$ for some $i' \leq 2$, this yields that $\overline{N(X_{i'})}$ is embeddable in $\overline{N(Y_j)}$. Except for the pair $(5, 3)$ (which has been

previously ruled out) this is clearly impossible. With this last argument, the proof is complete. □

4. Scattered distributive lattices

In this section, we consider *bounded* distributive lattices, that is distributive lattices with a least and a largest element denoted respectively by 0 and 1. If T is a such a lattice, an ideal I is *prime* if its complement $T \setminus I$ is a filter. The *spectrum* of T, that we denote $Spec(T)$, is the subset of $\mathfrak{P}(T)$ made of prime ideals of T. W.r.t. the topology on $\mathfrak{P}(T)$, this a closed subspace of $\mathfrak{P}(T)$, and with the inclusion order added, this is a Priestley space. The set of order preserving and continuous maps from $Spec(T)$ onto the two element chain $\underline{2}$ is a distributive lattice isomorphic to T. This fact is the essence of Priestley duality. We give below the facts we need in order to prove Theorem 4. We only give the proofs or an hint when needed. The first one is obvious:

Lemma 20. *If D is a chain with least and largest elements 0 and 1, then as a Priestley space, $Spec(D)$ is isomorphic to $\mathbf{I}(C)$ where $C := D \setminus \{0, 1\}$.*

Lemma 21. *Let T be a distributive lattice and C be a maximal chain of $Spec(T)$. Then, as a Priestley space, C is isomorphic to $Spec(D)$ where D a chain, quotient of T.*

For a proof, note that the spectrum of T, $Spect(T)$, is closed under unions and intersections of non-empty chains. Hence C is a complete chain.

We recall that the *width* of a poset P, denoted by $width(P)$, is the supremum of the cardinalities of the antichain of P. The following result is due to Dilworth [9].

Theorem 9. *Let T be a distributive lattice and n be an integer. Then $dim(T) \leq n$ if and only if $width(Spec(T)) \leq n$.*

Lemma 22. *Let T be a distributive lattice, two elements x, y of T such that $x < y$ and $T' := [x, y]$. Then $Spec(T')$ is isomorphic as a Priestley space to $A := \{J \in Spect(T) : x \in J \text{ and } y \notin J\}$.*

Proof. Let $\phi : A \to Spec(T')$ and $\theta : Spec(T') \to A$ defined by setting $\phi(I) := I \cap T'$ and $\theta(I') :=\downarrow I'$ are order preserving, continuous and inverse of each other. □

Lemma 23. *Let T be a distributive lattice. If $Spec(T)$ is not topologically scattered, there is some element $x \in T \setminus \{0, 1\}$ such that the spectra of $T'' :=\downarrow x$ and $T' :=\uparrow x$ are not topologically scattered.*

Proof. Since $Spec(T)$ is not topologically scattered, it contains a perfect subspace. Let P be such a subspace. Since $|P| \geq 2$, we may pick $J', J'' \in P$ such that $J' \not\subseteq J''$. Let $x \in J' \setminus J''$. Then $Spec(T'')$ and $Spec(T')$ are not scattered. Indeed, note first that according to Lemma 22, $Spec(T'') = \{J \in Spec(T) : x \notin J\}$ and $Spec(T') = \{J \in Spec(T) : x \in J\}$. Next, observe that the sets $F' := \{J \in P : x \in J\}$ and $F'' := \{J \in P : x \notin J\}$ are perfect. Since there are respectively contained in $Spec(T')$ and $Spec(T'')$ the conclusion follows. □

Theorem 10. *Let T be a distributive lattice. Then T is order-scattered if and only if $Spec(T)$ is topologically scattered.*

Proof. Suppose that $Spec(T)$ is not topologically scattered. For each pair of elements x, y in T such that $Spec([x, y])$ is not topologically scattered, Lemma 23 yields some $z \in]x, y[$ such that neither $Spec([x, z])$ nor $Spec([z, y])$ is topologically scattered. This fact allows to define an embedding ϕ from the set D of dyadic numbers of the $[0, 1]$ interval of the real line. Since $D := \{\frac{m}{2^n} : n \leq m \in \mathbb{N}\}$ is dense, T is not order scattered. Conversely, if T is not order scattered,select a non scattered chain and extend it to a maximal chain, say D. The natural embedding from D into T yields a continuous surjective map from $Spec(T)$ onto $Spec(D)$. As a Priestley space, $Spec(D)$ is isomorphic to $\mathbf{I}(C)$ where $C := D \setminus \{0, 1\}$ (Lemma 20). Since C is not order scattered, $Spec(D)$ is not topologically scattered. According to Rudin's result ((1) of Lemma 14), $Spec(T)$ is not topologically scattered. □

4.1. **Proof of Theorem 4.** We prove the result for bounded lattices. If T is not bounded, we add a least and a largest element, and apply the result to the resulting lattice. $(ii) \Rightarrow (i)$ Apply Proposition 5.
$(i) \Rightarrow (iii)$ Trivial.
For the proof of $(iii) \Rightarrow (ii)$, we introduce the following property:
(iv) *$Spec(T)$ is order scattered and $width(Spec(T)) \leq n$.*

We prove successively $(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (ii)$.
$(iii) \Rightarrow (iv)$. Suppose that (iii) holds. Since T is order scattered, Theorem 10 ensures that $Spec(T)$ is topologically scattered. With the inclusion order and the topology, $Spec(T)$ is a Priestley space, hence it is order scattered. Since $dim(T) \leq n$, Theorem 9 ensures that $width(Spec(T)) \leq n$. Thus, (iv) holds.
$(iv) \Rightarrow (ii)$. Suppose that (iv) holds. Cover $Spec(T)$ with m chains, where $m := width(Spec(T))$. Extend each of these chains to a maximal chain of $Spec(T)$. According to Lemma21, each maximal chain C_i is of the form $Spec(D_i)$ where D_i is a chain. Since $Spec(T)$ is order scattered, C_i and hence D_i is order scattered. Let $C := \oplus_{i<m} Spec(D_i)$ and $f : C \to Spec(T)$ defined by setting $f(x, i) := x$. The duality between distributive lattices and their Priestley spaces, yields a lattice embedding from T into $\Pi_{i<n} D_i$. Hence (ii) holds. □

5. Two-dimensional scattered posets

A linear extension L of an ordered set P is called *separating* if there are elements $x, y, z \in P$ with $x <_P z$, y incomparable with both x and z but $x <_L y <_L z$. Let P be an ordered set. If the order of P is the intersection of two non-separating linear extensions C and C' of P, C' is called a *complement* of C. Dushnik and Miller[11] gave the following characterization of ordered sets of dimension at most 2.

Theorem 11. *Let P be an ordered set, the following properties are equivalent:*
- *(i) $dim(P) \leq 2$.*
- *(ii) There is a linear extension of P which is non-separating.*
- *(iii) P is embeddable in the family of intervals of some chain, these intervals being ordered by inclusion.*
- *(iv) The incomparability graph of P is a comparability graph.*

We mention the following property.

Lemma 24. *Let P be a poset of dimension 2, L be a non-separating extension of P and I be an initial segment of L. For $z \in P$, we define $D(z) := (\downarrow z) \cap I$. Then for every $x, y \in P \setminus I$, with $x \parallel_P y$ we have:*

1) $D(x) \subseteq D(y)$ *or* $D(y) \subseteq D(x)$*,*
2) If $D(x) \subset D(y)$ *then* $y <_L x$*.*

Proof. 1) Suppose by contradiction that there are $u \in D(x) \setminus D(y)$ and $v \in D(y) \setminus D(x)$. Since L is a linear extension of P, I is an initial segment of P. Since $u, v \in I$, if $v \leq_P u$ then $v \in D(x)$ and if $u \leq_P v$ then $u \in D(y)$, a contradiction; so $u \parallel_P v$. With no loss of generality, we may suppose $u <_L v$. Since $u \leq_P v$, if $v <_L x$ then, since L is non-separating, we have $v \leq_P x$, a contradiction. Hence $x <_L v$. Since I is an initial segment of L we have $x \in L$, contradicting the hypothesis that $x \notin I$.

(2) Let $v \in D(y) \setminus D(x)$. Necessarily, $v <_P y$ and, since I an initial segment of L, $v \parallel_P x$. If $x \parallel_P y$ and $x <_L y$ then, since L is non-separating, we have $x <_L v$, which contradicts the fact that I is an initial segment of L. Hence, either $x \not\parallel_P y$, in which case $x <_P y$ or $y <_L x$. □

5.1. **The dyadic tree.** In the two-dimensional case we have:

Proposition 6. *If* T_2 *is embeddable in a product of two chains then both are non scattered.*

We deduce this from the following proposition.

Proposition 7. *Every non-separating extension of the dichotomic tree* T_2 *has order type* $\omega(1 + \eta)$*.*

Proof. We use the *condensation method* (see [24] pp. 71). Let $\mathcal{L}$ be a non-separating linear extension of the order on T_2 and L be the corresponding chain. Two elements $x, y \in T_2$ are *equivalent* if the interval they determine in L is finite. This is an equivalence relation. Each classe being an interval of L. The set of these equivalences is naturally ordered and the chain L is the lexicographical sum of these equivalence classes.

Claim 8. *Each equivalence class is a subchain of* T_2 *and has order type* ω*.*

Proof of Claim 8. We observe that for every $x \in T_2$, one of the two covers of x in T_2, namely $x0$ and $x1$, is a cover of x in L. Indeed, suppose for an example $x0 <_L x1$. If $x0$ is not a cover of x in L there is some y with $x <_L y <_L x0$. With respect to T_2 this element y is incomparable to x and $x0$ (if y was comparable to x we would have $x_1 \leq_{T_2} y$, whereas if y was incomparable to $x0$, then since T_2 is a tree, y would be comparable to x). Since $x <_{T_2} x0$, $\mathcal{L}$ is a separating linear extension, contradicting our hypothese. From this observation and the fact that $\downarrow x$ is finite for every $x \in T_2$, the claim follows. □

Claim 9. *The set* D *of equivalence classes has order type* $1 + \eta$*.*

Proof of Claim 9. Since T_2 has a least element, D too. Also D has no largest element. Otherwise, let X be the largest class. Pick $x \in X$. Since X is a subchain of T_2, one of the two covers of x is not in X; its equivalence class is larger than X, a contradiction. Finally, not class X has a cover in D. Otherwise, if Y is a cover of X, let y be the least element of Y. Since $\downarrow y$ is finite, there is $x \in X$ wich is incomparable to y (w.r.t. T_2). Let x' be the cover of x in T_2 which does not belong to X. We have $x <_{T_2} x'$, $x <_L y <_L < x'$, y incomparable to x and x' (w.r.t. T_2). This contradicts the fact that $\mathcal{L}$ is a non-separating extension. □

With these claims, the proof of Proposition 7 is complete. □

5.2. **Non-separating scattered extensions.** The "bracket relation"

$$\eta \to [\eta]^2_2 \tag{25}$$

a famous unpublished result of F.Galvin, asserts that if the pairs of rational numbers are divided into finitely many classes then there is a subset X of the rationals which is order-isomorphic to the rationals and such that all pairs being to the union of two classes (for a proof, see [29] Theorem 6.3 p.44 or [12] A.5.4 p.412 and, for a far reaching generalization, see [8]). This result expresses in a very economical way what the partitions of pairs look like. Indeed, what it really says is this:

Theorem 12. *Let $[\mathbb{Q}]^2$ be the set of pairs of rational numbers and $A_1, \dots, A_n$ be a partition of $[\mathbb{Q}]^2$. For every order $\leq_\omega$ on $\mathbb{Q}$ with order type ω there is a subset X of $\mathbb{Q}$ of order type η and indices i and j (with possibly $i = j$) such that all pairs of X on which the natural order on $\mathbb{Q}$ and the order $\leq_\omega$ coincide belong to A_i and all pairs of X on which these two orders disagree belong to A_j.*

The proof of Theorem 12 from (25) is immediate: intersect the partition $A_1, \dots, A_n$ with the partition U,V associated with the two orders (U being made of pairs on which the two orders coincide, and V being made of the other pairs) and apply iteratively the bracket relation to the resulting partition in order to find X whose pairs belong to the unions of two classes.

Partitions, or orders, associated to two linear orderings on the same set, like the natural order on the rational numbers and an order of type ω are called *sierpinskizations*. Clearly, $\Omega(\eta)$ is a sierpinskization of $\omega\eta$ with ω, whereas $\Omega(\eta)^*$ is a sierpinskization of $\omega\eta$ and ω^*. These two posets are the basic sierpinskizations of a non scattered chain with ω and ω^*. Indeed, if α and α' are two non scattered countable chains then their sierpinskization with ω are equimorphic (see [19] Corollary 3.4.2).

From Theorem 12, we have easily:

Proposition 8. *Let $P := (E, \leq)$ be a poset. If neither η, $\Omega(\eta)$ nor $\Omega(\eta)^*$ is embeddable in P then for every non scattered linear extension $\mathcal{L}$ of the order on P, and every subset $A \subseteq E$ such that $(A, \mathcal{L}_{\restriction A})$ has type η there is an antichain A' of P which is included in A and such that $(A', \mathcal{L}_{\restriction A'})$ has type η.*

Proof. Let $A \subseteq E$ such that $(A, \mathcal{L}_{\restriction A})$ has type η. Let A_1, resp. A_2, be the set of pairs $\{x, y\}$ of $[A]^2$ such that x and y are comparable, resp. incomparable (w.r.t. the order on P). Fix an order $\leq_\omega$ of type ω on A. Theorem 12 yields a subset A' of A and $i, j \in \{1, 2\}$ such that all pairs of A on which the order $\mathcal{L}$ and the order $\leq_\omega$ coincide belong to A_i and all pairs of A on which these two orders disagree belong to A_j. As it is easy to check, the three cases $i = j = 1$, $i = 1, j = 2$ and $i = 2, j = 1$ yield respectively that $P_{\restriction A'}$ is a chain of type η, contains a copy of $\Omega(\eta)$ and contains a copy of $\Omega(\eta)^*$. Thus these cases are impossible. The only remaining case $a = j = 2$ yields the desired conclusion. □

Theorem 13. *The following properties are equivalent:*

(i) P is the intersection of two scattered chains.

(ii) (a) P has a non separative scattered extension and

(b) Neither $\Omega(\eta)$ nor $\Omega^*(\eta)$ is embeddable in P.

Proof. $(i) \Rightarrow (ii)$. Item (ii) (a) follows from Theorem 11. Item (ii)(b) follows from the fact that $\Omega(\eta)$ is an obstruction. $(ii) \Rightarrow (i)$. Let $\mathcal{C}$ be a non-separative

scattered extension of P. Let $\mathcal{C}'$ be the complement of $\mathcal{C}$. To conclude, it suffices to prove that $\mathcal{C}'$ is scattered. Suppose that it is not. Apply Proposition 8 to P and $\mathcal{L} := \mathcal{C}'$. Clearly, neither η, $\Omega(\eta)$ nor $\Omega^*(\eta)$ is embeddable in P. Thus, there is an antichain A' of P such that $C'_{\restriction A'}$ has type η. But, since A is an antichain of P and the order on P is the intersection of $\mathcal{C}$ and $\mathcal{C}'$, it turns out that $\mathcal{C}_{\restriction A}$ is the dual of $\mathcal{C}'_{\restriction A}$, thus $(A', \mathcal{C}_{\restriction A'})$ has type η. This contradicts the fact that $\mathcal{C}$ is scattered. □

Bibliographical comments. The posets T_2, $\Omega(\eta)$, $B(\check{\eta})$ have been considered previously. Pouzet and Zaguia [19] proved that *the set $\mathcal{J}(P)$ of ideals of a poset P contains no chain isomorphic to η if and only if P contains no chain isomorphic to η and no subset isomorphic to $\Omega(\eta)$*. In [10] it is shown that *if a poset P contains $B(\check{\eta})$, then $N(P)$ contains a chain isomorphic to η*. In fact, $N(B(\check{\eta}))$ is isomorphic to the disjoint union $\mathbb{Q} \times 2 \cup I(\mathbb{Q})$ equipped with the following ordering:

(1) $\emptyset \leq (x, 0) \leq I \leq (y, 1) \leq \mathbb{Q}$ for $x \in I \subseteq (\leftarrow y[$, with $I \in I(\mathbb{Q})$, $y \in \mathbb{Q}$.
(2) $I \leq J$ if $I \subseteq J$ and $I, J \in I(\mathbb{Q})$.

In [21] it is shown that *the class of posets whose MacNeille completion is scattered is characterized by eleven obstructions*. One can check that obstructions distinct from η and $B(\check{\eta})$ do no yield interesting obstructions to $\mathcal{L}_{\mathcal{S}}(< n)$. In an unpublished paper with E.C.Milner [17] it is shown that if the set $\mathcal{J}(P)$ of ideals of a poset P is topologically closed in $\mathfrak{P}(P)$, it is topologically scattered if and only if it is order scattered and the binary tree T_2 is not embeddable in P. From this follows that *an algebraic lattice T is topologically scattered if and only if it is order scattered and neither T_2 nor $\Omega(\eta)$ are embeddable in the join-semilattice of compact elements of T*. In contrast, we may note that *an algebraic distributive lattice is topologically scattered if and only if it is order scattered*, an important result due to Mislove [18].

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